# Airy Line ensemble

**Goal**:- Motivate and define Airy line ensemble.

A bit of random matrices :-

**Physics** :- Wigner introduced random symmetric/Hermitian matrices to study energy spectrum of heavy nuclei. Interested in energy spectral gaps.

**Statistics** :- Wishart studied sample covariance of iid sample of a population.

# Gaussian Unitary Ensemble (GUE):-

$$M_n = (\xi_{ij})_{i,j=1,2,\dots n}$$

$\xi_{ij} = \xi_{ji}^*$ - complex conjugate.

$\xi_{ij} \sim N_{\mathbb{C}}(0,1)$ $\quad i \neq j$

$\xi_{ii} \sim N_{\mathbb{R}}(0,1)$.

and entries of $M_n$ are independent save for the condition $M_n^\dagger = M_n$.

Density of $M_n$ w.r.t. Lebesgue :-

$$\mu_{M_n} = C_n \left( \prod_{1 \le i < j \le n} e^{-|\xi_{ij}|^2} \right) \left( \prod_{i \le n} e^{-\frac{|\xi_{ii}|^2}{2}} \right) dM_n$$

$$= C_n \, e^{-\frac{tr(M_n^2)}{2}} \, dM_n$$

For a deterministic unitary $U$,

$$Tr\left[ (UMU^\dagger)^2 \right] = Tr(UM^2U^\dagger) = Tr(M^2)$$

Hence $\mu_{M_n}$ is invariant under unitary transformations, hence the name GUE.

<u>Ginibre formula</u> (density of e.v.) :-

$$\boxed{\rho_n(\lambda_1, \dots \lambda_n) = C_n \cdot |\Delta_n(\lambda)|^2 \, e^{-\sum_{i=1}^{n} \lambda_i^2/2}}$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are eigenvalues of $M_n$ (GUE) and

$$\Delta_n(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

$\lambda = (\lambda_1, \lambda_2 \dots \lambda_n)$ is Vandermond determinant.

"<u>Proof</u>" :- Write $M_n = U D_n U^{+}$

This representation is unique upto

(i) permutation of diagonal entries of $D_n$

(ii) $U \mapsto \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} U$

So $\left(\mathbb{R}^n / S_n\right) \times \left(U(n) / \mathbb{T}^n\right) \xleftrightarrow{\text{bijection.}} \mathbb{R}^{n^2}$

$\downarrow$ Hermitian matrices

$\mathbb{T}$ - circle

$S_n$ - permutation group

Let $\lambda_1, \lambda_2, \cdots \lambda_n, \omega_{11} \cdots \omega_{n(n-1)}$

be the local coordinates of

$\left(\mathbb{R}^n / S_n\right) \times \left(U(n) / \mathbb{T}^n\right)$.

Want to compute Jacobian of

$(\lambda_1, \lambda_2 \cdots \lambda_n, \omega_1 \cdots \omega_{n(n-1)}) \mapsto$

$(\xi_{11}, \xi_{22} \cdots, Re(\xi_{12}), \cdots)$

we get that

$$\text{Jacobian} = |\Delta_n(\lambda)|^2 \times \text{fun}^{\underline{c}} \text{ of } \omega_1, \omega_2, \ldots$$

Hence, $U$ and $D_n$ are independent!

Since GUE has the measure

$$C_n' \, e^{-\frac{1}{2} \operatorname{Tr}(M_n^2)} \, dM_n$$

and noting $\operatorname{Tr}(M_n^2) = \sum_{i=1}^{n} \lambda_i^2$, $\boxed{\text{change of variables}}$

we have the density for e.v.

$$\rho_n(\lambda) = C_n \cdot |\Delta_n(\lambda)|^2 \cdot e^{-\frac{1}{2}\sum \lambda_i^2}$$

∎

Exercise:- Do the computation for 2x2 case i.e. $M_2$.

Remarkably distribution of $U$ (unitary) in $M_n = U D_n U^{+}$ is uniform, i.e. $U \overset{d}{=} \text{Haar}(U(n))$.

↳ group of unitary matrices.

This is the magic of Gaussianity !!

For $Z \sim N_{\mathbb{C}}(0,1)$, note that

$|Z|$ and $\arg(Z)$ are independent

and $\arg(Z)$ is uniform.

<u>Physical intuition</u>:- Think of e.v.

$\lambda_1, \lambda_2 \dots \lambda_n$ as charged particles

in a quadratic potential well.

i.e. $$-\log \rho_n(\lambda) = \frac{1}{2} \|\lambda\|_2^2 + \sum_{i \neq j} \log \frac{1}{|\lambda_i - \lambda_j|} + C.$$

<u>Determinantal point-processes</u>:-

Fix a set $S$ (say $\mathbb{R}^2$ or $\mathbb{R}$).

Let $K: S \times S \to \mathbb{R}$ be positive semi-definite function, called a <u>kernel</u>.

(Informal) A (simple) point process on $S$ is a determinantal point process with kernel $K$ if

$$dP(x_1, \ldots x_n) = c_n \cdot \det\left(K(x_i, x_j)\right)_{i,j \le n} \times \prod_{k=1}^{n} dx_k$$

Coming back to e.v. density of GUE,

Observe,

$$\Delta_n(\lambda) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & & & & \vdots \end{pmatrix}$$

called Vandermonde indentity.

By column operations,

$$\Delta_n(\lambda) = \det \begin{pmatrix} p_0(\lambda_1) & p_1(\lambda_1) & \cdots & p_{n-1}(\lambda_1) \\ p_0(\lambda_2) & \cdot & & \\ \vdots & & \ddots & \\ p_0(\lambda_n) & & & p_{n-1}(\lambda_n) \end{pmatrix}.$$

for any monic polynomial $p_k$ of degree $k$.

Hence,

$$P_n(\lambda) = C_n \cdot \det \begin{pmatrix} p_0(\lambda_1)\, e^{-\lambda_1^2/4} & p_1(\lambda_1) e^{-\lambda_1^2/4} & \cdots \\ p_0(\lambda_2)\, e^{-\lambda_2^2/4} & & \\ \vdots & & \ddots \end{pmatrix}^2$$

$$= c''_n \det(A^T A)$$

where $A_{ij} = p_{j-1}(\lambda_i)\, e^{-\frac{1}{4}\lambda_i^2}$.

Take $\{p_k(x)\}_{k\geq 0}$ to be a system of orthogonal polyomials w.r.t. measure $e^{-x^2/2}dx$. i.e.

$$\int_{\mathbb{R}} p_i(x)\, p_j(x)\, e^{-x^2/2}\, dx = \varkappa_i^2\, \delta_{ij}.$$

<u>Example</u> :- Hermite polynomials

$$p_k(x) = (-1)^k\, e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}.$$

In this case, $\varkappa_j^2 = j!\sqrt{2\pi}$.

Define $\phi_j(x) := \varkappa_j^{-1}\, p_j(x)\, e^{-x^2/4}$

(Hermite func$^{\underline{c}}$)

$\{\phi_j\}_{j\geq 0}$ are orthonormal basis of $L^2(\mathbb{R})$.

Hence, $\rho_n(\lambda) = C_n'' \cdot \det(A^T A)$

$$(A^T A)_{ij} = \sum_{k=1}^{n} A_{ki} A_{kj} = \sum_{k=0}^{n-1} \phi_k(\lambda_i)\phi_k(\lambda_j)$$

$$K_n(x,y) := \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{1}{k!} P_k(x) P_k(y) e^{-\frac{(x^2+y^2)}{4}}$$

Hence, $(\lambda_1, \lambda_2 \ldots \lambda_n)$ is a determinantal point process with kernel $K_n$.

## Wigner's semi-circle law:-

For $M_n$ GUE matrix, spectral radius = $L^2$-matrix norm

Hence, $v \in \mathbb{R}^n$, $\|v\|_2 = 1$,

$$\mathrm{Var}\left(\|M_n v\|_2^2\right) \sim n$$

To get meaninful scaling for the eigenvalues, need to scale by $\sqrt{n}$.

**Theorem**:- Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded continuous $\mathrm{fun}^{\underline{c}}$ (test $\mathrm{fun}^{\underline{c}}$). Then,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i/\sqrt{n}) = \int_{\mathbb{R}} f(x)\, d\sigma(x)$$

$$d\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\, dx \; \mathbb{1}[|x| \leq 2].$$

Convergence in expectation and almost surely.

Think of point process as random measure

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{(\lambda_i/\sqrt{n})} \longrightarrow \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

<u>Spacing of GUE eigenvalues</u>:-

Microscopic repulsion:-

<u>Exercise</u>:- $n=2$, $\lambda_1, \lambda_2$ e.v. of $M_2$, then $|\lambda_1-\lambda_2|$ has density $s^2 e^{-s^2/8}$,

$\Rightarrow$ density vanishes at $s=0$ (repulsion)

Similar density for $|\lambda_j-\lambda_{j+1}|$ for $n>2$. In fact, "any" Hermitian random matrix has quadratic repulsion

Since $d\sigma$ has uniformly positive density inside $[-2,2]$, spacing between eigenvalues is roughly $1/\sqrt{n}$.

## Scaling at the edge of the spectrum:-

By semi-circle law,

$$\#\{\lambda_i : \lambda_i/\sqrt{n} \geq 2-\varepsilon\} \simeq \frac{n}{2\pi}\int_{2-\varepsilon}^{2} \sqrt{4-x^2}\,dx$$

$$\simeq \frac{2}{3\pi} n \varepsilon^{3/2}$$

To see finite number of e.v.,

correct scale is $\varepsilon \simeq n^{-2/3}$.

i.e. $\dfrac{\lambda_i}{\sqrt{n}} = 2 + \alpha_i n^{-2/3}$

$\Rightarrow \lambda_i = 2\sqrt{n} + \alpha_i n^{-1/6}$.

Hence, the correct scaling to get a non-trivial limit is,

$$\tilde{K}_n(x,y) = n^{-1/6} K_n(2\sqrt{n} + x n^{-1/6}, 2\sqrt{n} + y n^{-1/6})$$

Lemma:- $\forall x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left| Ai(x) - n^{1/12} \phi_n(2\sqrt{n} + x n^{-1/6}) \right| = 0$$

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(t^3/3 + xt)\, dt$$

Airy function.

$\phi_n$ - $n^{th}$ Hermite fun$\underline{c}$.

Thm:- $\lim_{n\to\infty} \tilde{K}_n(x,y) =$

$$\frac{Ai(x)\, Ai'(y) - Ai'(x)\, Ai(y)}{(x-y)}$$

Summary:- At the edge of the spectrum,

$$\lambda_i = 2\sqrt{n} + \alpha_i n^{-1/6} \quad \text{(location)}$$

Spacing is roughly $n^{-1/6}$

(Compared to $n^{-1/2}$ in the bulk).

To get constant mean spacing,

consider $\{n^{1/6} \lambda_i\}_{i=1,\dots,n}$, so

location of $n^{th}$ particle is of

order $n^{2/3}$.

We define stationary Airy line ensemble.

Def$^n$ :- Extended Airy kernel

$\mathcal{K}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is given by,

$$\mathcal{K}(s, x; t, y) = \int_0^{\infty} e^{u(t-s)} Ai(x+u) Ai(y+u)\, du$$

if $s \geq t$.

$$\mathcal{K}(s, x; t, y) = -\int_{-\infty}^{0} e^{u(t-s)} Ai(x+u) Ai(y+u)\, du.$$

if $s < t$.

The stationary Airy line ensemble

$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots) \in \mathbb{Z}_{\geq 1} \times C(\mathbb{R})$

is a collection of random continuous curves $\mathcal{A}_i : \mathbb{R} \to \mathbb{R}$, $\mathcal{A}_1(t) > \mathcal{A}_2(t) > \dots$

$\forall t \in \mathbb{R}$ such that the density

$(t_1, y_1), (t_2, y_2) \dots (t_m, y_m) \in \mathcal{A}$ is

$$\det\left[ K(t_i, y_i ; t_j, y_j) \right]_{i,j \le n} \prod_{j=1}^{m} dy_j$$

<u>Remarks</u>:- ① Put $t = s$ in the kernel to get Airy point process.

② Since $Ai(x) \simeq \frac{1}{x^{1/4}} \exp\left(-\frac{2}{3} x^{3/2}\right)$

the density of points decay exponentially, hence a.s there is a "top" point / curve.

③ Existence of such process:- see Corwin-Hammond [2014]

Scale and location of Airy point process :-

* Forget about eigenvalues scaling.

Want to study $(A_1(0), A_2(0), \cdots)$

One-point correlation / density of points of determinantal point process with kernel $K$ is

$$\mathbb{E}\left[\#\text{ pts. in } I\right] = \int_I K(x,x)\,dx$$

$I \subseteq \mathbb{R}$ (say).

For Airy point process,

$$K(x,x) = -x\left(Ai(x)\right)^2 + \left(Ai'(x)\right)^2$$

Using the asymptotics of Airy function, we have

$$\mathbb{E}\left[\#\text{ pts in } (-T, \infty)\right] = \frac{2}{3\pi} T^{3/2} + O(1)$$

as $T \to \infty$.

and,

$$\mathbb{E}\left[A_j(0)\right] = -\left(\frac{3\pi}{2} \cdot j\right)^{2/3} + O(1).$$

Fine location:-

$$\mathbb{P}\left(\left|A_j(0) + \left(\frac{3\pi}{2} j\right)^{2/3}\right| \geq m j^{-1/3}\right) \leq c e^{-\frac{m}{5}}$$

for $m \geq c_1 \log j$

Proof is out of scope for us.

(Determintal str. of Airy pt. process.)

Now think of each line $\mathcal{A}_j$ as stationary perturbation of $\left(\frac{3}{2}\pi j\right)^{2/3}$.

<u>Definition</u>:- parabolic Airy line ensemble $\mathcal{R} := (\mathcal{A}_1(t) - t^2, \mathcal{A}_2(t) - t^2, ---)$

[Scaled version] $S = 2^{-1/2} \mathcal{R}$.

Hence we have, with high probability on paths.

$$S_j(t) + 2^{-1/2} t^2 = -2^{-7/6} (3\pi)^{2/3} j^{2/3} + O(j^{o(1) - 1/3})$$

Eq (1.2) in Aggarwal - Huang

If $j \simeq n$ and $t \simeq n^{1/3}$ then

$S$ is of order $n^{2/3}$ in the box

$[-n^{1/3}, n^{1/3}] \times [-n^{2/3}, 0]$ $(t, x)$-space

Plan :- Compare the Brownian Gibbsian line ensemble $\mathcal{L}$ to $S$ in the above box scale.

[Brownian motion characterisation of Airy line ensemble].

- Airy point process $\leftrightarrow$ scaling of e.v. of GUE at the edge.

- Airy lines $\longleftrightarrow$ stationary perturb-
-ation of Airy points.

<u>Dyson Brownian motion</u>:- $N \in \mathbb{N}$.

Let $B_{ij}(t)$ be independent complex Brownian motion, $i<j$.

For $B_{ii}(t)$ be standard real BM.

All of them start at zero.

$$A(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) & \cdots \\ \overline{B_{12}(t)} & B_{22}(t) & \\ \vdots & & \end{pmatrix}$$

Then the eigenvales of $A(t)$ evolve according to the sol$^n$ of

$$\boxed{d\lambda_i = dB_i + \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j}} \quad (\#)$$

where $B_i$'s are iid real BM.

Fact:- As $N \to \infty$, the top $k$ curves $\lambda_1(t) > \lambda_2(t) > \cdots \lambda_k(t)$ (after rescaling) converge to Airy line ensemble.

[Ornstein-Uhlenbeck entries $\Rightarrow$ Stationary Airy line ensemble.]

pictures.

## Non-intersecting Brownia motions:-

Take $N$ independent real BM

$X_1, X_2, \ldots X_N$ and condition

it to never collide, say

$X_1(t) > X_2(t) \cdots > X_N(t)$.

By Doob's h-transform,

the new generator of the process

$$\mathcal{L}^h(f) = \frac{1}{h}\mathcal{L}(hf)$$

where $h(x) = \Delta(x) = \prod_{i<j}(x_i - x_j)$

and $\mathcal{L}(f) = \frac{1}{2}\sum_{i=1}^{N} \partial_{x_i}^2 f$.

The induced drift is

$$\nabla \log h(x) = \left( \sum_{j \neq i} \frac{1}{x_i - x_j} \right)_{i=1}^{N}$$

$$\Rightarrow dX_i(t) = dB_i(t) + \sum_{j \neq i} \frac{dt}{X_i(t) - X_j(t)}$$

By uniqueness of strong soln of SDE, $X_i$'s are Dyson BM.

In conclusion, Airy line ensemble is the ensemble of top curves of non-intersecting Brownian motions(!!)