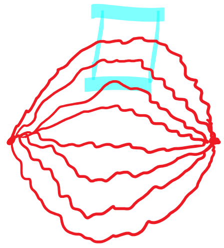
Proof overview of Main Theorem (Strong Characterization)



Brownian watermelon
(completely solvable and well studied and well behaved)

$\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \dots)$ parabolic Airy line ensemble

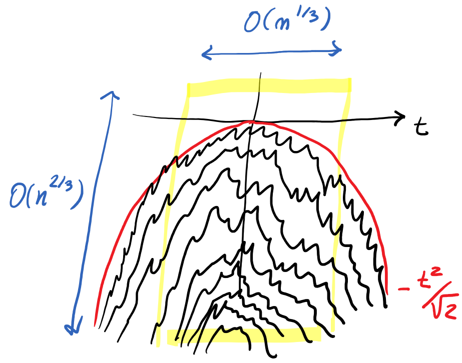
$\mathcal{S} = \frac{1}{\sqrt{2}}\mathcal{R}$ scaled parabolic ALE, which satisfies the Brownian Gibbs property with standard BM

$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2 \dots)$ generic Brownian Gibbsian line ensemble

satisfying $\mathbb{P}\left[\, \left| \mathcal{L}_1(t) + \frac{t^2}{\sqrt{2}} \right| \le \varepsilon t^2 + C_\varepsilon \right] \ge 1-\varepsilon$ $\forall t$ for all ε fixed

and for $C_\varepsilon > 0$ depending only on ε.

Some properties of $\mathcal{S}$:



$$(\star_1) \qquad \mathcal{S}_j(t) = -\frac{t^2}{\sqrt{2}} - \frac{(3\pi)^{2/3}}{2^{7/6}} j^{2/3} + O(j^{o(1)-1/3})$$

with high probability (Sodnikov '00).

As a result taking $t = O(n^{1/3})$ and $j = O(n)$, we have

$$\mathcal{S}_j(t) = O(-n^{2/3})$$

and calling $f_n(t) = -\frac{t^2}{\sqrt{2}} - \frac{(3\pi)^{2/3}}{2^{7/6}} n^{2/3}$ we have

$$(\star_2) \qquad \left| \mathcal{S}_n(t) - f_n(t) \right| = o(1) \qquad \text{with high probability}$$

Notice that f_n is smooth in time.

Moreover, fixing $g_r(y) = -\frac{(3\pi)^{2/3}}{2^{7/6}}(1+y)^{2/3} - \frac{r^2}{\sqrt{2}}$, we have

$$(\star_3) \qquad \left| \mathcal{S}_{n+k}(t) - n^{2/3} g_{t/n^{1/3}}(k/n) \right| = o(1) \qquad \text{with high probability.}$$

Strategy: Establish analogs of $(\star_1)$, $(\star_2)$, $(\star_3)$ for the line ensemble $\mathcal{L}$.

4 macro-themes :

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4 macro-Themes:

(1) On-scale estimates: Fix $A>1$ large and $t\in[-An^{1/3}, An^{1/3}]$ and $j\in[1\dots An]$

-1.1) Path locations: with high probability we have

$$-450\, j^{2/3} \le \mathcal{L}_j(t) + \frac{t^2}{\sqrt{2}} \le -\frac{j^{2/3}}{200} \qquad \forall j \ge \frac{n}{A} \qquad \forall |t|\le An^{1/3}$$

- 1.2) Gap upper bound: with high probability, for $i\le j$:

$$|\mathcal{L}_i(t) - \mathcal{L}_j(t)| = O(i^{2/3} - j^{2/3}) + \frac{(\lg n)^{25}}{i^{1/3}}$$

(we expect $= O(i^{2/3} - j^{2/3})$)

(2) Global law and regularity:

- Global law: We have

$$\left|\mathcal{L}_j(t) + \frac{t^2}{\sqrt{2}} + \frac{(3\pi)^{2/3}}{2^{7/6}} j^{2/3}\right| = o(n^{2/3}) \qquad (\star_1')$$

- Spatial regularity: there exists with high probability a (potentially random) function $\gamma_r : [0,1] \to \mathbb{R}$ (analog of $g_r(y)$) such that

$$|\mathcal{L}_{n+k}(t) - n^{2/3}\gamma_{t/n^{1/3}}(k/n)| = o(1) \qquad \forall k\in[1\dots n] \qquad (\star_3')$$

(3) Curvature approximation: there likely exists a potentially random function $h_n : [-An^{1/3}, An^{1/3}] \to \mathbb{R}$ so that

$$|h_n''(t) + \sqrt{2}| = o(1) \qquad = \frac{d^2}{dy^2}\frac{y^2}{\sqrt{2}}$$

$$(\star_2') \qquad |\mathcal{L}_n(s) + h_n(s)| = o(1) \qquad \forall s\in[-An^{1/3}, An^{1/3}].$$

(4) Airy statistics:

- Airy gaps: $(\mathcal{L}_1 - \mathcal{L}_2, \mathcal{L}_2 - \mathcal{L}_3, \dots)$ has the same law as $(\mathcal{S}_1 - \mathcal{S}_2, \mathcal{S}_2 - \mathcal{S}_3, \dots)$

- Airy line ensemble: $\mathcal{L}(t) = \mathcal{S}(t) + X_1 + X_2 t$ for X_1, X_2 random and independent.

these coefficients $(3\pi)^{2/3}$ will eventually random and independent.

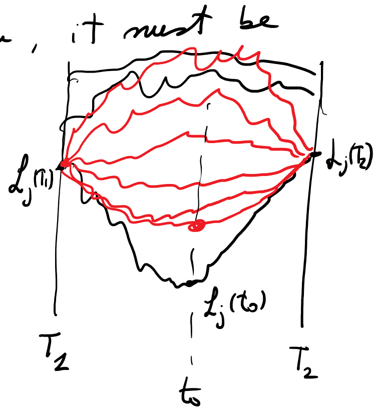
these coefficients will eventually be $= -\frac{(3\pi)^{2/3}}{2^{7/6}}$

1- On scale estimates:

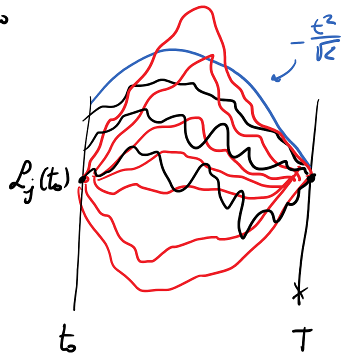
We need to show that whp $\quad -450\, j^{2/3} < \mathcal{L}_j(t) + \frac{t^2}{\sqrt{2}} < -\frac{j^{2/3}}{200}$

- Both location bounds follow from 3 ideas (using height monotonicity)

1) If $\exists\, t_0$ such that $\mathcal{L}_j(t_0)$ is too low, then, it must be also low on a large interval $[T_1, T_2] \ni t_0$.
If not resample $\mathcal{L}_1 \dots \mathcal{L}_{j-1}$ on $[T_1, T_2]$ and use height monotonicity (the black curves should stay above the red curves)



2) If $\mathcal{L}_j$ is too high at t_0, then it must be low on a large segment $[t_0, T]$. If not resample $\mathcal{L}_1 \dots \mathcal{L}_j$ with endpoint $\mathcal{L}_j(t_0)$, $\mathcal{L}_j(T)$ and prove that the top curve will necessarily stay above the parabola $-\frac{t^2}{\sqrt{2}}$



3) $\mathcal{L}_j$ cannot be too low on any long interval $[T_1, T_2]$. Otherwise you could resample the top $j-1$ curves with initial data $\mathcal{L}_i(T_1) = -\frac{T_1^2}{\sqrt{2}}$ $(\sim \mathcal{L}_1(T_1))$ and $\mathcal{L}_i(T_2) = -\frac{T_2^2}{\sqrt{2}}$ for $i = 1 \dots j-1$, in such a way that their top curve stays below the parabola $-\frac{t^2}{\sqrt{2}}$.