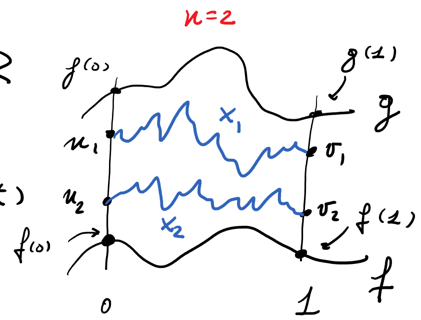
Last time: Monotonic couplings of line ensembles

For n-tuples of ordered numbers $u=(u_1>\dots>u_n)$, $v=(v_1>\dots>v_n)$ and two functions $f,g:[0,1]\to\mathbb{R}$ with $f(0)<u_n<\dots<u_1<g(0)$ and $f(1)<v_n<\dots<v_1<g(1)$ define the probability measure

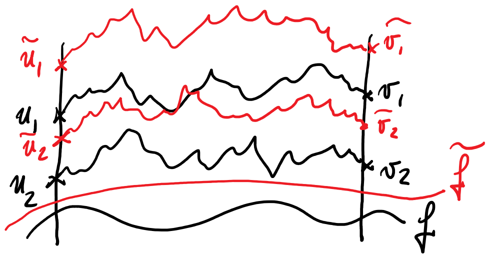
$$Q^{u,v}_{f,g} = \left\{ \begin{array}{l} n\text{-tuples of Brownian bridges } x_1\dots x_n:[0,1]\to\mathbb{R} \\ \text{such that } x_i(0)=u_i,\ x_i(1)=v_i \text{ and conditioned} \\ \text{to } f(t)<x_n(t)<\dots x_{i+1}(t)<x_i(t)<\dots x_1(t)<g(t) \\ \forall t\in[0,1] \end{array}\right\}$$



If $g=\infty$ we write $Q^{u,v}_f$.

Height monotonicity coupling: given $f,\tilde f:[0,1]\to\mathbb{R}$ such that $f(t)<\tilde f(t)$ and n-tuples $u,\tilde u,v,\tilde v$ such that $u_i<\tilde u_i$ and $v_i<\tilde v_i$ $\forall i\in\{1\dots n\}$ we can couple $x\sim Q^{u,v}_f$ and $\tilde x\sim Q^{\tilde u,\tilde v}_{\tilde f}$ so that

$$x_i(t)\le \tilde x_i(t) \qquad \forall\ i\in\{1..n\},\ t\in[0,1]$$



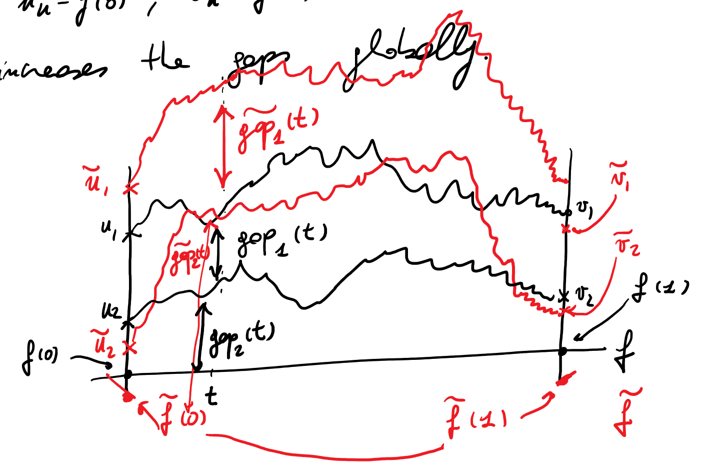
• $= \tilde x_i(t)$
• $= x_i(t)$

Gap monotonicity coupling: In words: increasing gaps of boundary points u_i-u_{i+1}, v_i-v_{i+1} and $u_n-f(0)$, $v_n-f(1)$ and increasing the convexity of boundary function f increases the gaps globally.

$0\le u_n-f(0)\le \tilde u_n-\tilde f(0)$

$0\le u_i-u_{i+1}\le \tilde u_i-\tilde u_{i+1}$

(and same for $f(1)$, $\tilde f(1)$, v, $\tilde v$)

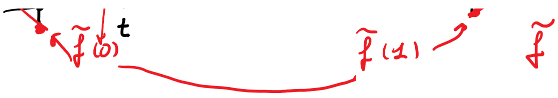


The coupling between x, $\tilde x$ is such that

$0\le x_i(t)-x_{i+1}(t)\le \tilde x_i(t)-\tilde x_{i+1}(t)$

and $0\le x_n(t)-f(t)\le \tilde x_n(t)-\tilde f(t)$

$v, \tilde v$)



$\le \tilde x_n - \tilde f(t)$

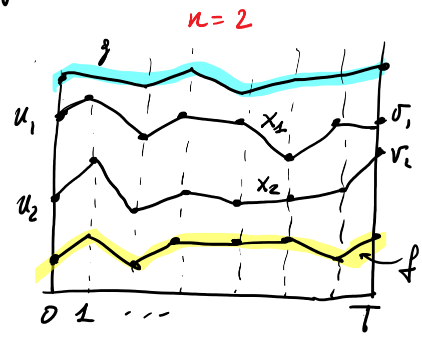
$\mathrm{gap}_i(t) \le \widetilde{\mathrm{gap}}_i(t) \quad \forall i, t$

$\forall t \in [0,1]$ and $i \in \{1 .. n-1\}$

Idea: (semi-) discretize. Spoiler: gap-monotonicity does not a priori generalize to discrete random walks

Definition Fix $T \in \mathbb{N}$. A T-steps Gaussian random walk is a random walk with independent increments with law $\mathcal{N}(0,1)$.
A T-steps Gaussian random walk bridge is a T-steps Gaussian random walk conditioned to have specific endpoints.
Fix $n \in \mathbb{N}$. Given functions $f, g : \{0, \dots, T\} \to \mathbb{R}$ and n-tuples of numbers u, v such that $f(0) < u_n < \dots < u_1 < g(0)$ and $f(T) < v_n < \dots < v_1 < g(T)$
we define the probability measure

$$\mathcal{G}_{f,g}^{u,v} = \left\{ \begin{array}{l} \text{n-tuples of T-steps Gaussian RWB } X = (X_1 .. X_n) \\ \text{such that } X_i(0) = u_i,\ X_i(T) = v_i \text{ and for all} \\ t \in \{0 .. T\} \quad f(t) < X_n(t) < \dots < X_1(t) < g(t) \end{array} \right\}$$



As before, if $g = \infty$ we use the notation $\mathcal{G}_f^{u,v}$

Proposition (Semi-discrete Gap Monotonicity : AH'25)
Fix ordered n-tuples $u, \tilde u, v, \tilde v$ and functions $f, \tilde f : \{0 .. T\} \to \mathbb{R}$ such that

$$0 \le u_n - f(0) \le \tilde u_n - \tilde f(0), \quad 0 \le v_n - f(T) \le \tilde v_n - \tilde f(T)$$

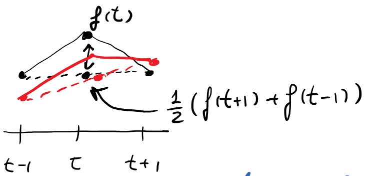
(gap with bottom boundary increased)

$$0 \le u_i - u_{i+1} \le \tilde u_i - \tilde u_{i+1}, \quad 0 \le v_i - v_{i+1} \le \tilde v_i - \tilde v_{i+1}$$

(gaps between boundary values increased)

$\forall i \in \{1 .. n-1\}$. Assume that, $\forall t \in \{1, \dots, T-1\}$

$$f(t+1) - 2f(t) + f(t-1) \le \tilde f(t+1) - 2\tilde f(t) + \tilde f(t-1)$$



$$f(t) - \frac{1}{2}[f(t+1) + f(t-1)] \ge \tilde f(t) - \frac{1}{2}[\tilde f(t+1) + \tilde f(t-1)]$$

(increased convexity of bottom boundary)

Then, there exists a coupling between $X \sim \mathcal{G}_f^{u,v}$ and $\tilde X \sim \mathcal{G}_{\tilde f}^{\tilde u, \tilde v}$ such that

Then, there exists a coupling of $\;$ of $\tilde{f}$
that

$$0 \le X_n(t) - f(t) \le \tilde{X}_n(t) - \tilde{f}(t)$$

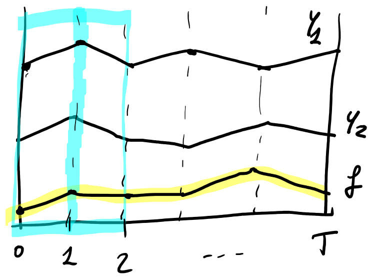
and $\quad 0 \le X_i(t) - X_{i+1}(t) \le \tilde{X}_i(t) - \tilde{X}_{i+1}(t) \quad \forall i \in \{1 \dots n-1\} \quad \forall t \in \{0 \dots T\}$

The proof is articulated into several steps

1. Prove the case $T=2$ (technical construction)

2. Induction over T with $T=2$ as base case.

Here the idea is to construct the coupling between T-steps ensembles $X \sim \mathcal{G}_f^{u,v}$, $\tilde{X} \sim \mathcal{G}_{\tilde{f}}^{\tilde{u},\tilde{v}}$ by using coupling for ensembles of 2-steps and $(T-1)$-steps



Definition (Alternating dynamics) Fix $T, n \in \mathbb{N}$, $f : \{0 \dots T\} \to \mathbb{R}$ and non intersecting paths $y = (y_1 > \dots > y_n)$ $\quad y_i : \{0 \dots T\} \to \mathbb{R}$. Define the random collection of non-intersecting T-steps walks

$$P^k y(t) = \left(P^k y_1(t) > \dots > P^k y_n(t) > f(t) \right)$$

as follows

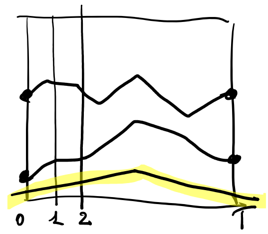
- if $k=0$ $\quad P^k y = y$
- if k is odd $\quad$ for $t \in \{2, \dots T\}$ $\quad P^k y_j(t) = P^{k-1} y_j(t) \quad \forall j$

  and for $t=1$ we sample $P^k y_j(t) \sim \mathcal{G}_{f|_{\{0,1,2\}}}^{P^{k-1}y_j(0),\, P^{k-1}y_j(2)}$
- if k is even, for $t=1$ $\quad P^k y_j(1) = P^{k-1} y_j(1) \quad \forall j$

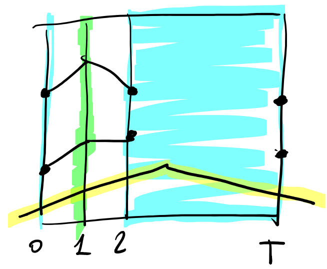
$P^{k-1}y(1), P^{k-1}y(T)$

• if k is even, for $t = 1$ $P y_j(1) = P\, y_j(1)$ $\forall j$

and for $t \in \{2 \dots T\}$ we sample $\left(P^k y_j(t)\right)_{t \in \{1 \dots T\}} \sim \mathcal{G}_f^{P^{k-1} y(1),\, P^{k-1} y(T)}\Big|_{\{1 \dots T\}}$



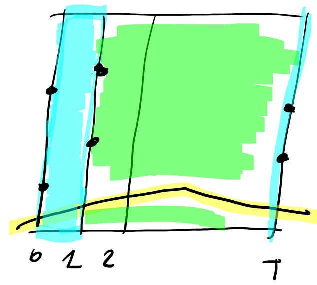
initial condition



k odd

= sample

= frozen



k even

= sample

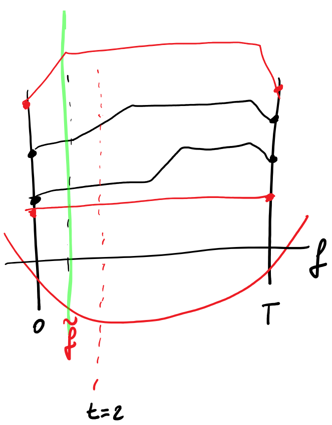
= frozen

<u>Lemma</u> For any initial condition y the random dynamics $P^{2k} y$ converges to $\mathcal{G}_f^{y(0),\, y(T)}$ (in total variation distance) for $k \to \infty$

<Idea of proof of Semi discrete gap monotonicity models $T = 2$ case>

The alternating dynamics gives a way to sample $\mathcal{G}_f^{u,v}$ of T-steps non-intersecting RWB using the measure $\mathcal{G}_f^{u,v}$ of 2-steps NIGRWB and $(T-1)$-steps NIGRWB. By induction over T we can construct gap monotone coupling of 2-steps NIGRWB and $(T-1)$-steps NIGRWB.

Now we have to show that using these couplings in the alternating dynamics produces a gap monotone coupling also for the T-step NIGRWB.



<u>Strategy</u>: Fix $u, \tilde{u}, v, \tilde{v}$ $f, \tilde{f}$ as in the hypotheses of the proposition (see picture). Construct initial paths $y, \tilde{y}$ such that $y_i(0) = u_i$, $y_i(T) = v_i$, $\tilde{y}_i(0) = \tilde{u}_i$, $\tilde{y}_i(T) = \tilde{v}_i$.

Run the alternating dynamics where at each step k you couple $(P^k y(t))\big|_{t \in \{0,1,2\}}$, $(P^k \tilde{y}(t))\big|_{t \in \{0,1,2\}}$

so that they are gap monotone and [illegible]

$t=2$

. . . .

for k odd so to have gap monotonicity and similarly for k even on the complementary t.

It is clear that this procedure creates ensembles $\mathcal{P}^k_{\mathcal{Y}}$ and $\mathcal{P}^k_{\tilde{\mathcal{Y}}}$ whose gaps are increasing, i.e.

$$0 \le \mathcal{P}^k_{\mathcal{Y}_i}(t) - \mathcal{P}^k_{\mathcal{Y}_{i+1}}(t) \le \mathcal{P}^k_{\tilde{\mathcal{Y}}_i}(t) - \mathcal{P}^k_{\tilde{\mathcal{Y}}_{i+1}}(t) \quad \forall t, k, i.$$

Taking $k \to \infty$ completes the proof.

$\langle$End of video of proof, modulo case $T=2\rangle$