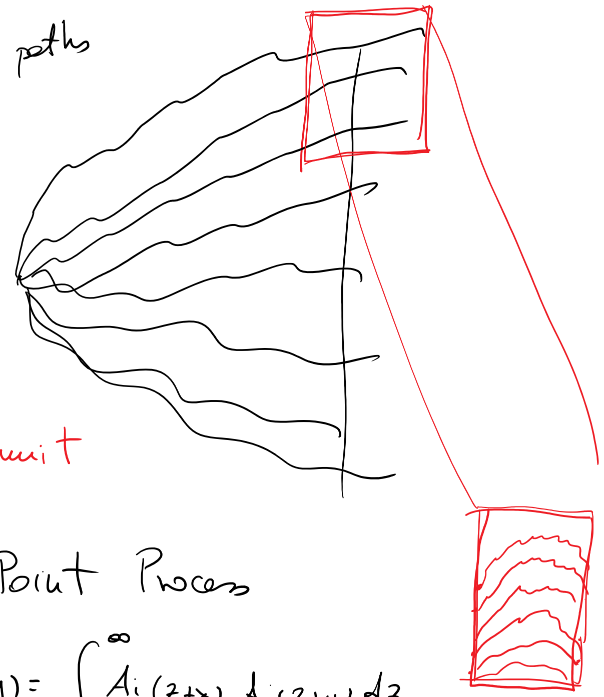
Goal: take a scaling limit around the first few paths



Under such scaling limit:

1) DPP is preserved

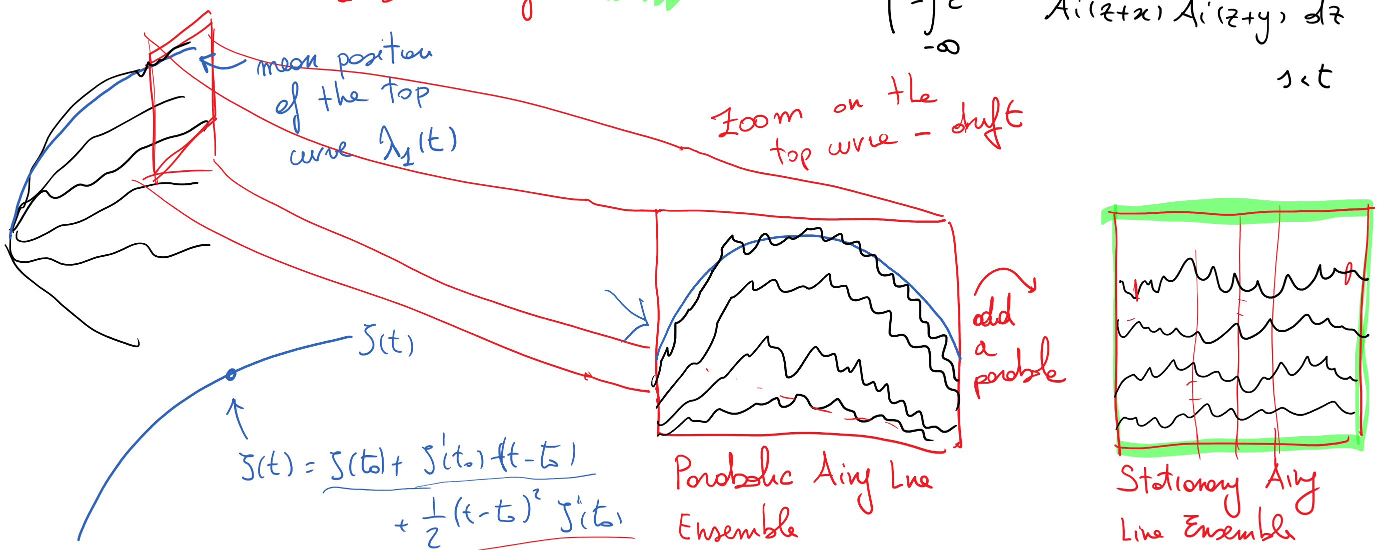
2) Brownian Gibbs property is preserved

Finite object		Scaling limit
GUE	$\longrightarrow$	Airy Point Process

$$K_{N,\mathrm{Hermite}}\left(2\sqrt{N}+\frac{x}{N^{1/6}},\ 2\sqrt{N},\ \frac{y}{N^{1/6}}\right) \longrightarrow K_{\mathrm{Airy}}(x,y)=\int_0^\infty Ai(z+x)\,Ai(z+y)\,dz$$

DBM

$$K^{\mathrm{ext}}_{N,\mathrm{Hermite}}(\dots) \xrightarrow[\text{1:2:3 scaling}]{} K^{\mathrm{ext}}_{\mathrm{Airy}}(t,x;s,y)=\begin{cases}\displaystyle\int_0^\infty e^{-z(s-t)}\,Ai(z+x)\,Ai(z+y)\,dz & s\geq t\\[2ex] \displaystyle-\int_{-\infty}^0 e^{-z(s-t)}\,Ai(z+x)\,Ai(z+y)\,dz & s<t\end{cases}$$

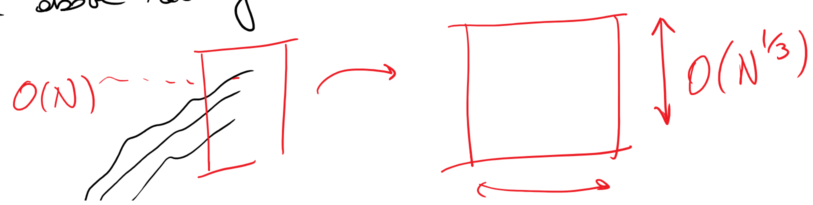


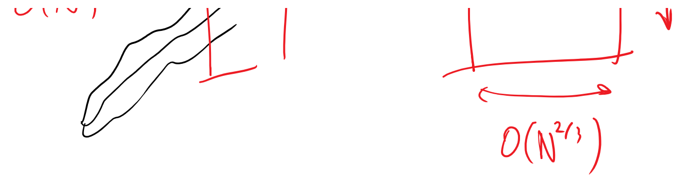
$$\zeta(t)=\zeta(t_0)+\zeta'(t_0)(t-t_0)+\frac{1}{2}(t-t_0)^2\,\zeta''(t_0)+\dots$$

Parabolic Airy Line Ensemble $\downarrow$ Has the Brownian Gibbs property

Stationary Airy Line Ensemble $\downarrow$ Is an extended DPP with correlation kernel $K^{\mathrm{ext}}_{\mathrm{Airy}}$

1:2:3 scaling means that the above rectangle is

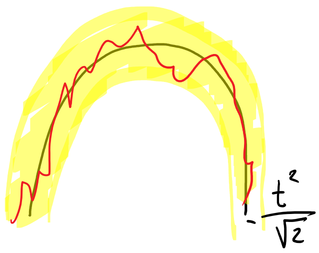




So far: we understand the definition and the origin of the (Parabolic / stationary) Airy Line Ensemble.

Theorem (Aggarwal-Huang '23) Let $\mathcal{L} = (\mathcal{L}_1 \geq \mathcal{L}_2 \geq \dots)$ be a Brownian Gibbsian Line Ensemble such that $\forall \varepsilon > 0 \ \exists\ C_0 : \forall\ c > C_0$, we have

$$\mathbb{P}\left[-\left(\frac{1}{\sqrt{2}} + \varepsilon\right) t^2 - c \leq \mathcal{L}_1(t) \leq -\left(\frac{1}{\sqrt{2}} - \varepsilon\right) t^2 + c \right] \geq 1 - \varepsilon .$$

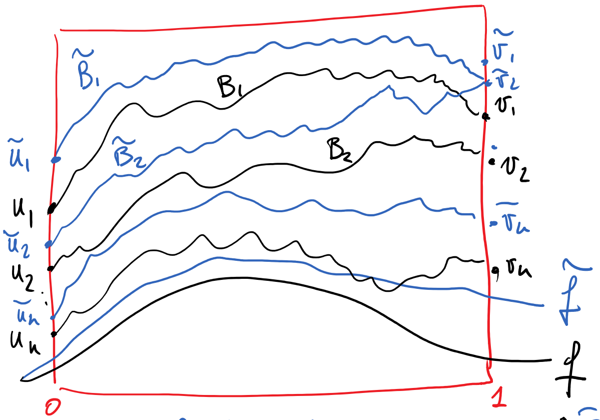


Then $\forall j \geq 1$ $\mathcal{L}_j(t) = \mathcal{S}_j(t) + t\mathcal{X}_1 + \mathcal{X}_2$ where $\mathcal{X}_1, \mathcal{X}_2$ are random variables and $\mathcal{S} = (\mathcal{S}_1 \geq \mathcal{S}_2 \geq \dots)$ is the parabolic Airy Line Ensemble. Moreover $\mathcal{X}_1, \mathcal{X}_2$ are indep. from $\mathcal{S}$.

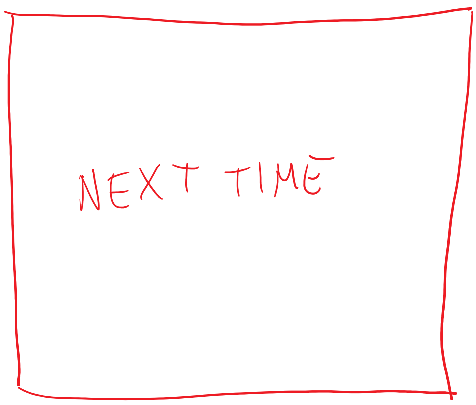
Today: Some coupling arguments.

Basic coupling results

1) Height monotonicity



2) Gap monotonicity



$u = (u_1 \geq \dots \geq u_n)$, $\tilde{u} = (\tilde{u}_1 \geq \dots \geq \tilde{u}_n)$ $v = (v_1 \geq \dots v_n)$, $\tilde{v} = (\tilde{v}_1 \geq \dots \geq \tilde{v}_n)$ are endpoints of n non-intersecting Brownian Bridges $B_1 \geq \dots \geq B_n$: $B_n(t) \geq f(t)$ and $\tilde{B}_1 \geq \dots \geq \tilde{B}_n$: $\tilde{B}_n(t) \geq \tilde{f}(t)$.

If $u_i \le \tilde{u}_i$ and $v_i \le \tilde{v}_i$ and $f(t) \le \tilde{f}(t)$
we can couple $B, \tilde{B}$ so that

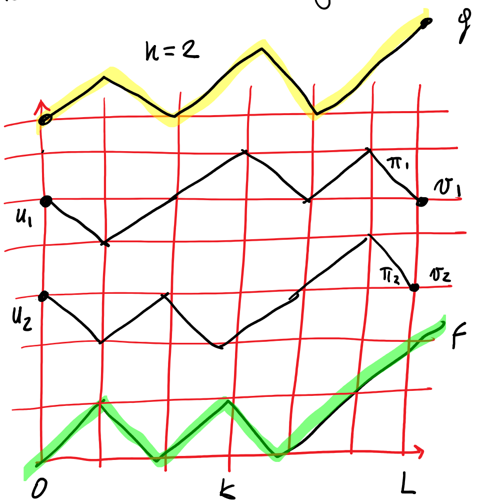
$B_i(t) \le \tilde{B}_i(t) \quad \forall i = 1, \dots, n \quad t \in (0,1)$

Proof of Height Monotonicity (Constructive)

Idea: discretize : Brownian Bridges $\rightsquigarrow$ Random Walk Bridges

Consider $u = (u_1 \ge \dots \ge u_n) \in \mathbb{Z}^n$ and $v = (v_1 \ge \dots \ge v_n) \in \mathbb{Z}^n$
and $f, g : \{0, \dots, L\} \to \mathbb{Z}$, Consider the set

$$W_{f;g}^{\vec{u},\vec{v}} = \Big\{ n\text{-tuples of paths } (\pi_1, \dots, \pi_n) : \{0, \dots, L\} \to \mathbb{Z}^n \text{ such that } \pi_i(0) = u_i,\ \pi_i(L) = v_i,\ \pi_i(k) > \pi_{i+1}(k)\ \forall i = 1 \dots n-1,\ \pi_1(k) < g(k) \text{ and } \pi_n(k) > f(k)\ \forall k = 0, \dots L : \text{ moreover } \pi_i(k) - \pi_{i+1}(k) \in \{\pm 1\} \Big\}$$

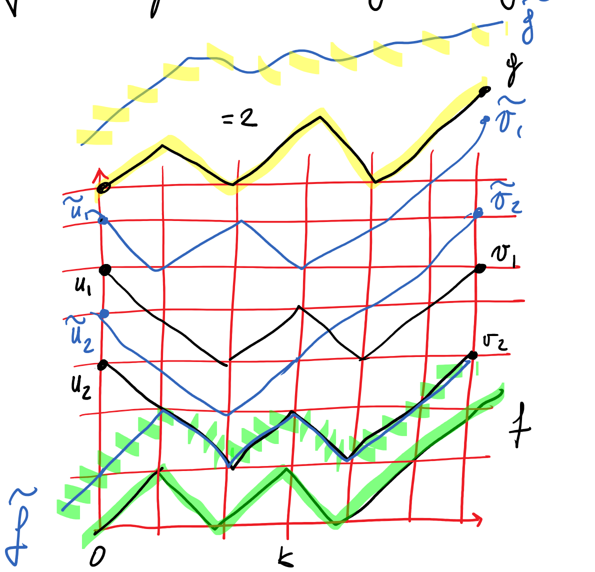


a pair of walks in $W_{f,g}^{u,v}$

The set $W_{f,g}^{u,v}$ is finite and we sample random n-tuples of walks π uniformly.

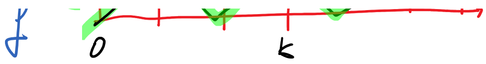
Prop (discrete variant of Height monotonicity)

Let $u, \tilde{u}, v, \tilde{v}$ be n-tuples of integers such that $u_i \le \tilde{u}_i$, $v_i \le \tilde{v}_i$. Fix $f, \tilde{f}, g, \tilde{g} : \{0, \dots, L\} \to \mathbb{Z}$ such that $f(k) \le \tilde{f}(k)$ and $g(k) \le \tilde{g}(k)$ $\forall k \in \{0, \dots, L\}$. Then, we can couple n-tuples of paths $\pi \in W_{f;g}^{u,v}$ and $\tilde{\pi} \in W_{\tilde{f},\tilde{g}}^{\tilde{u},\tilde{v}}$ so that $\pi \sim \mathrm{Unif}(W_{f;g}^{u,v})$, $\tilde{\pi} \sim \mathrm{Unif}(W_{\tilde{f};\tilde{g}}^{\tilde{u},\tilde{v}})$ and $\pi_i(k) \le \tilde{\pi}_i(k)$ $\forall i = 1 \dots n$ and $k = 0, \dots, L$

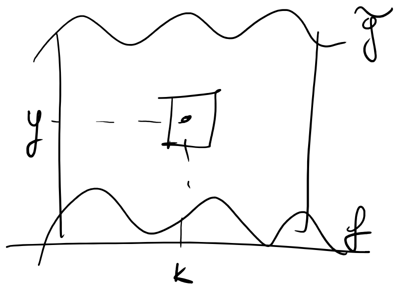


<proof>

We sample $\pi, \tilde{\pi}$ symultaneously using

We sample $\pi, \tilde{\pi}$ symultaneously using 
glauber dynamics. Pick any pair $\pi^{(0)}, \tilde{\pi}^{(0)}$ satisfying $\pi_i^{(0)}(k) \le \tilde{\pi}_i^{(0)}(k)$.

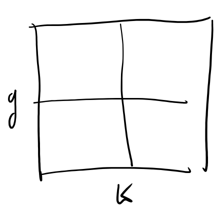
- Pick (k, y) : the point $k \in \{1 \dots L-1\}$ and $y \in \{ f(k), \tilde{g}(k) \}$



- if (k,y) is not a local min for either π_i or $\tilde{\pi}_i$ for some i, move on.

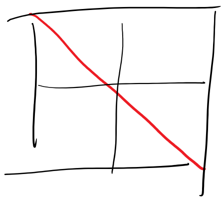
- if (k,y) is a local min for π_i, or $\tilde{\pi}_i$ for some i flip download the paths lying local min there

- same for (k,y) local max



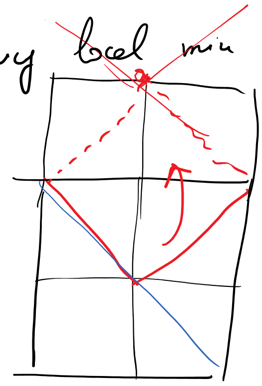
no path

Do Nothing



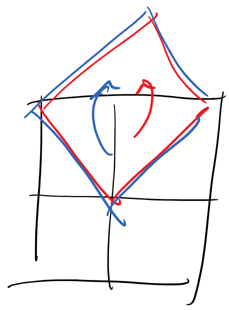
there are paths but no local min or max

Do Nothing



there is a path lying local min at (k,y)

Flip and turn k into a local max



If the updated paths belong to $W_{f,g}^{u,v}$, $W_{\tilde{f},\tilde{g}}^{\tilde{u},\tilde{v}}$, accept the update otherwise no.

Observation: Such algorithm is monotonic: if the initial paths $\pi^{(0)}, \tilde{\pi}^{(0)}$ are ordered, then, they will stay ordered at all times.