Last time:

GUE random matrices. Fix $N \in \mathbb{N}$

$$M = (M_{ij})_{i,j=1}^{N}$$

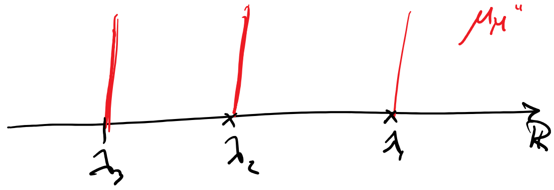
random Hermitian Gaussian matrix

$$M_{ij} = \begin{cases} N(0,\tfrac{1}{2}) + i\,N(0,\tfrac{1}{2}) & i<j \\ N(0,1) & i=j \\ \overline{M_{ji}} & i>j \end{cases}$$

Theorem (Semicircle Law)
Define the empirical measure of the spectrum of M, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$
(the eigenvalues are real, by hermitianity)

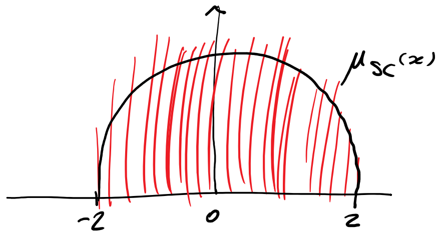
$$\mu_M(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(x)$$

"plot of μ_M"



Then, we have

$$\mu_M(\sqrt{N}\,x) \xrightarrow{\text{weakly a.s.}} \mu_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+} \qquad (z)_+ = \begin{cases} z & \text{if } z \geq 0 \\ 0 & \text{else} \end{cases}$$



Theorem (Hermite DPP)
Let $\lambda = (\lambda_1 \dots \lambda_N)$ be the spectrum of a GUE matrix.

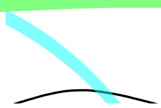
$$K_{N,\text{Hermite}}(x,y) = \phi_i(x)\,\phi_i(y) \quad \left(= \frac{\phi_N(x)\,\phi_{N-1}(y) - \phi_{N-1}(x)\,\phi_N(y)}{x-y} \right)$$

Christoffel-Darboux identity

where $\phi_i(x)$ are the (L^2-normalized) Hermite polynomials, which are an L^2-orthogonal basis of $L^2(\mathbb{R}, dx)$.

This means that (informally)

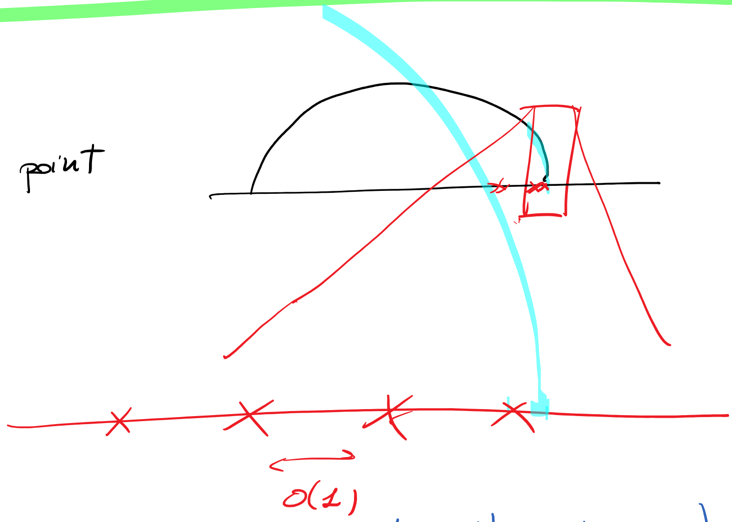
$$\mathbb{P}\left[\begin{array}{l}\text{there is at least one eigenvalue} \\ \text{in the disjoint infinitesimal intervals} \\ (x_1, x_1+dx_1) \cup (x_2, x_2+dx_2) \cup \dots \cup (x_k, x_k+dx_k)\end{array}\right] = \det\left[K_{N,\text{Hermite}}(x_i, x_j)\right]_{i,j=1}^{k} dx_1 \dots dx_k$$



The Sine Limit

Edge Scaling limit

This can be determined from the DPP point of view,



We have

$$K_{N,\text{Hermite}}\left(2\sqrt{N}+\frac{x}{N^{1/6}},\ 2\sqrt{N},\ \frac{y}{N^{1/6}}\right)$$

$$\xrightarrow[N\to\infty]{} K_{Airy}(x,y) = \int_0^\infty Ai(z+x)\,Ai(z+y)\,dz \quad \left(= \frac{Ai(x)\,Ai'(y)-Ai'(x)\,Ai(y)}{x-y}\right)$$

Dyson's Brownian Motion: We can add a dimension to the above analysis. (Idea: $N(0,\sigma^2) \longmapsto B_t(0,\sigma^2)$)

Definition: Fix $N\in\mathbb{N}$

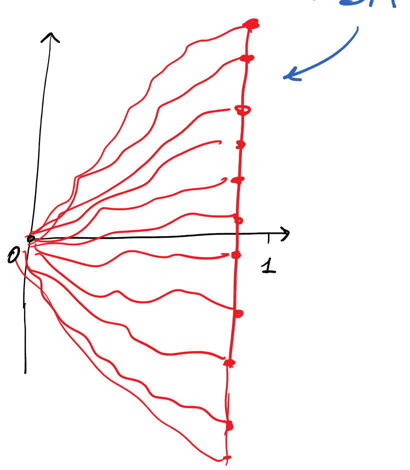
$$M(t) = \left(M_{ij}(t)\right)_{i,j=1}^{N}$$

each entry i,j gets 2 indep. BM

$$M_{ij}(t) = \begin{cases} B_t(0,\tfrac12) + i\tilde{B}_t(0,\tfrac12) & i<j \\ B_t(0,1) & i=j \\ \overline{M_{ji}(t)} & i>j \end{cases}$$

Once again, since for each t $M(t)$ is Hermitian, $\lambda(t) = (\lambda_1(t) \geq \dots \geq \lambda_N(t)) \subset \mathbb{R}$. Then, the eigenvalues form a time dependent stochastic process, whose law at each fixed time is a rescaling of the law of the spectrum of a GUE matrix.

Picture



Theorem (Dyson's BM)

The process $\vec{\lambda}(t) = (\lambda_1(t) \geq \dots \geq \lambda_N(t))$ evolve according to the SDE

$$d\lambda_i(t) = \sum \frac{dt}{\lambda_i(t)-\lambda_j(t)} + dB_t^{(i)}$$

$$d\lambda_i(t) = \sum_{1\le i<j\le N} \frac{dt}{\lambda_i(t)-\lambda_j(t)} + dB_t$$

where $(B_t^{(i)})_{i=1,\dots,N}$ are independent BM.

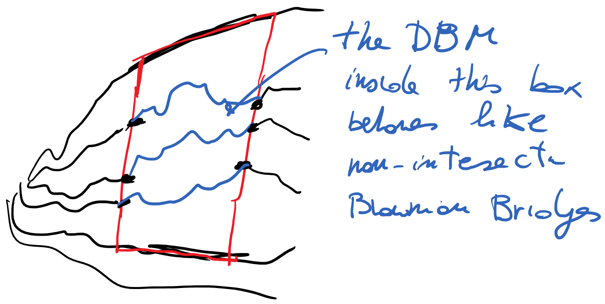
Observation: We can write the above SDE as

$$d\vec{\lambda}(t) = \nabla_\lambda \log\Big(\prod_{i<j}(\lambda_i-\lambda_j)\Big)\,dt + d\vec{B}_t$$

From the above expression one can recognize that λ is the "Doob's h-transform" of N independent Brownian motions, where $h=h(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j)$ is the harmonic function encoding the infinitesimal probability of the N Brownian motions to never intersect.

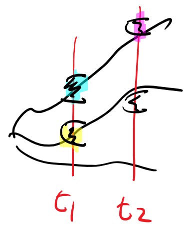
Theorem $\vec{\lambda}(t)$ has the law of N Brownian motions started at 0 and condition to never intersect.

Corollary: The DBM has the Brownian Gibbs property



Theorem The DBM is an extended DPP. Fix $t_1,\dots,t_k$, then

$$\mathbb{P}\Big(\text{there is a path crossing the following intervals } \{t_1\}\times(x_1,x_1+dx_1),\ \dots,\ \{t_k\}\times(x_k,x_k+dx_k)\Big)$$



$$= \det\Big[K^{ext.}_{N,\mathrm{Hermite}}(x_i,t_i,x_j,t_j)\Big]_{i,j=1}^{k} dx_1\dots dx_k$$

Explicit generalization of $K_{N,\mathrm{Hermite}}$.