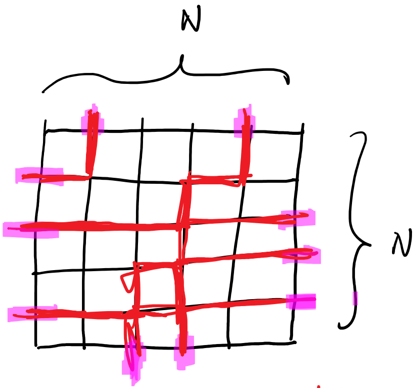
Strong Characterization for the Airy Line Ensemble

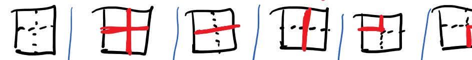
Today: Motivate the Airy Line Ensemble through the lens of the Ice Model

Ice Model (Pauling '35): A classical model in statistical mechanics originally introduced to understand the entropy of ice.



- $N \times N$ grid $\mathbb{L}_N$
- Ice configurations: configurations of up-right paths on $\mathbb{L}_N$ not sharing edges
- Fix some boundary conditions: entry locations of paths on the left and bottom boundary and exit locations of paths on the top and right boundary

Boltzmann Weights (local vertex configurations)



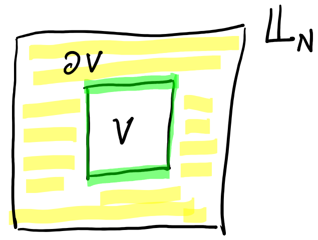
	e_1	e_2	e_3	e_4	e_5	e_6
ICE	1	1	1	1	1	1

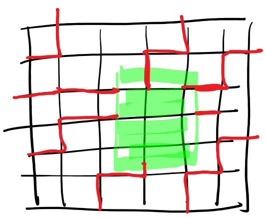
- Energy of global configuration $\mathcal{C}$ = $\mathcal{E}(\mathcal{C}) = \sum_{v \in \mathbb{L}_N} e(\mathcal{C}_v)$ ← configuration at the vertex v
- Gibbs measure: $\mathbb{P}(\mathcal{C}) = \dfrac{e^{-\mathcal{E}(\mathcal{C})}}{Z_N}$ $\quad Z_N = \sum_{\mathcal{C}} e^{-\mathcal{E}(\mathcal{C})}$ = partition function

Ice model $\quad \mathbb{P}(\mathcal{C}) = 1/Z_N \quad$ Z_N = # of configurations

Gibbs property (spatial Markov property)

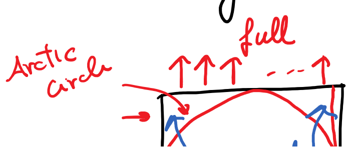
$$\boxed{\mathbb{P}\left((\mathcal{C}_v)_{v \in \mathring{V}} \,\middle|\, (\mathcal{C}_v)_{v \in \mathbb{L}_N \setminus V} \right) = \mathbb{P}\left((\mathcal{C}_v)_{v \in \mathring{V}} \,\middle|\, (\mathcal{C}_v)_{v \in \partial V} \right)}$$



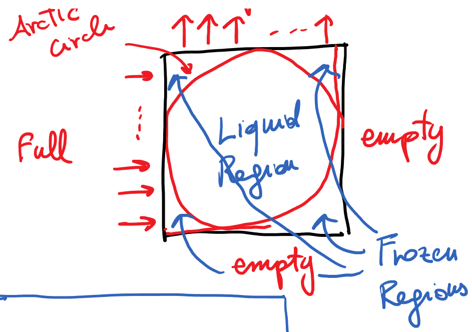


We consider a special class of boundary conditions:

Domain-Wall Boundary Conditions (DWBC)



It was observed by Syljuåsen-Zvonarev '04 & Allison-Reshetikhin '05 that



It was observed by Syljuåsen-Zvonarev '04 & Allison-Reshetikhin '05 that the Six Vertex Model (which is the Ice model with more general Boltzmann weights) exhibits the Arctic Circle Phenomena. The Arctic Circle Phenomenon was proven (only for Ice Model) by Aggarwal in '18.

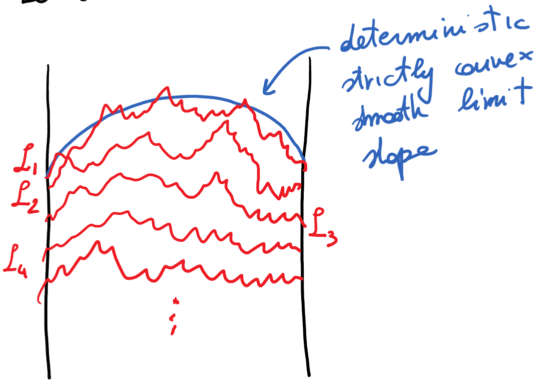
History of ACP.
- Domino Tiling (Cohn-Kenyon-Propp)
- maybe more refs. later

The Arctic Circle Phenomenon suggests co-existence of 2 phases: liquid phase inside the Arctic Circle and frozen phase outside of the Arctic circle.

As of today still little is known about fluctuations in the DWBC Ice Model. Conjecturally in the liquid region fluctuations should be described by the 2D Gaussian Free Field (this result has been announced by H. Duminil Copin - K. Kozlowski, ...). In the frozen region there are no fluctuations.

It remains to understand the boundary: the boundary process is (conjecturally still) given by the Airy Line Ensemble.

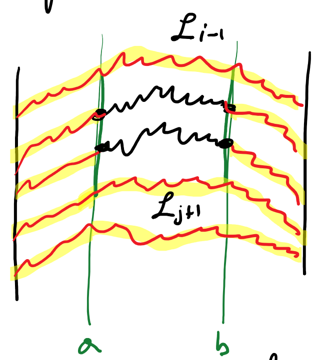
Let's examine some basic properties of the process at boundary.



0) Scaling at the boundary we will observe a family $\mathcal{L} = (\mathcal{L}_1 \geq \mathcal{L}_2 \geq \dots)$ of curves

1) Gibbs property:

Conditioned the values of paths at a and b and conditioned to paths $\mathcal{L}_1 \dots \mathcal{L}_{i-1}, \mathcal{L}_{j+1}, \mathcal{L}_{j+2} \dots$ paths $\mathcal{L}_i \dots \mathcal{L}_j$ in $[a,b]$ behave like Brownian bridges (scaling of random walks conditioned to not intersect and to not cross $\mathcal{L}_{i-1}$ and $\mathcal{L}_{j+1}$



2) Convex limit slope ⇒ taking Taylor expansion the top curve behaves like − a parabola.

2) Convex limit shape ⇒ [illegible] the top curve behaves like $-$ a parabola.

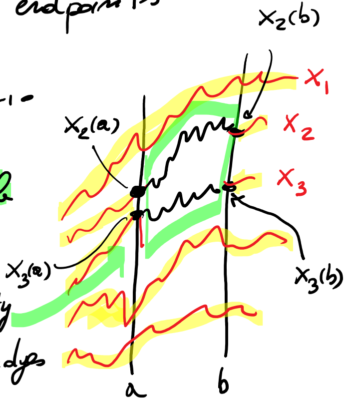
Definition (Brownian Gibbsian Line Ensemble)

Let $x = (x_1, x_2, \dots) : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ be a family of random functions $x_i(t)$ such that

1) they are ordered $x_1(t) > x_2(t) > \dots$

2) For any $1 \le i \le j$ and $a < b$ conditioned to $(x_k)_{k \notin \{i,\dots,j\}}$ and conditioned to $(x_k|_{\mathbb{R} \setminus [a,b]})_{k \in \mathbb{N}}$, the law of $(x_k|_{[a,b]})_{k \in \{i,\dots,j\}}$ is that of $j-i$ non intersecting Brownian bridges (standard) with endpoints $(x_k(a))_{k=i,\dots,j}$ and $(x_k(b))_{k=i,\dots,j}$, such that $x_j > x_{j+1}$ and $x_i < x_{i-1}$.

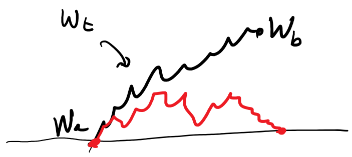
Then we say that x is a Brownian Gibbsian Line Ensemble



Comment: A Brownian bridge (standard) is $(B_t)_{t \in [a,b]}$

$$\underline{B_t} = W_t - \frac{t-a}{b-a} W_b$$

where $(W_t)_{t \in [a,b]}$ is a standard Brownian motion.

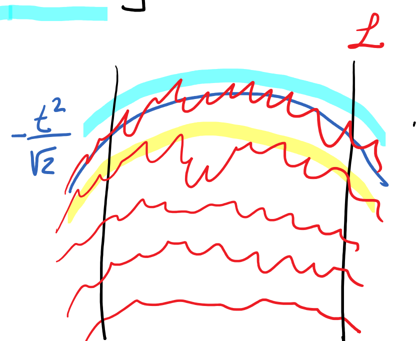


$B_a = 0 \quad B_b = 0$

Theorem (Aggarwal-Huang '23)

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \dots)$ be a Brownian Gibbsian Line Ensemble. Assume that for all $\varepsilon > 0$, there exists C_ε such that $\forall C > C_\varepsilon$ we have

$$\mathbb{P}\left[\underline{-(2^{-1/2} + \varepsilon)t^2 - C} \le \mathcal{L}_1(t) \le \underline{-(2^{-1/2} - \varepsilon)t^2 + C} \right] \ge 1 - \varepsilon.$$



Then, there exist two random variables X_1, X_2 such that

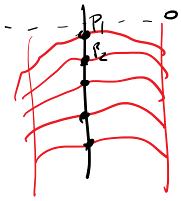
$$\boxed{\mathcal{L}_j(t) = \mathcal{S}_j(t) + t X_1 + X_2}$$

where $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2 \dots)$ is the Airy Line Ensemble (defined next time)

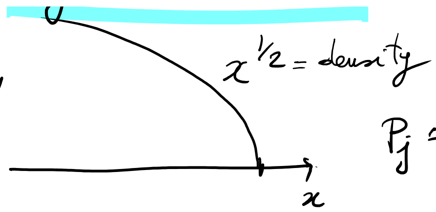
histogram of [illegible]

$x^{1/2} = $ density

Question:



histogram of



$P_j \simeq O(j^{-2/3})$